IV.

On a Short Process for Solving the Irreducible Case of Cardan's Method.

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The equation having three commensurable roots, a, b, c, is

$$x^3 - (a+b+c)\,x^2 + (ab+ac+bc)\,x - abc = 0\,.$$

Reducing the roots of this equation by $\frac{1}{3}(a+b+c)$, we have

$$y^3 - \frac{1}{3}(a^2+b^2+c^2-ab-ac-bc)\,y$$
$$-\frac{1}{27}(2a^3+2b^3+2c^3-3a^2b-3a^2c-3ab^2-3ac^2-3b^2c-3bc^2+12abc)=0.$$

This being of the form $y^3+my+n=0$, we have, substituting in Cardan's formula and reducing,

$$y=\frac{1}{3}\sqrt[3]{-\frac{27}{2}n+\frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3}}$$
$$+\frac{1}{3}\sqrt[3]{-\frac{27}{2}n-\frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3}}=u+v\,.$$

Since u is a binomial imaginary, its cube root will be of the form $\alpha+\sqrt{-\beta}$, and $\sqrt{\frac{\beta}{3}}$ will be rational. Hence

$$(1)\qquad \alpha(\alpha^2-3\beta)=\frac{1}{2}(2a^3+2b^3+2c^3-3a^2b-3a^2c-3ab^2-3ac^2$$
$$-3b^2c-3bc^2+12abc)\,,$$

$$(2)\qquad \sqrt{-\beta}\,(3\alpha^2-\beta)=\frac{3}{2}(a-b)(a-c)(b-c)\sqrt{-3}=(r).$$

Since α must be rational, $\sqrt{-\beta}$ must be of the first degree with reference to a, b, c, and the only factors of (r) of that degree are of the form $\frac{(a-b)\sqrt{-3}}{p}$, where p is some integer: p must be 2, for substituting $\sqrt{-\beta}=\frac{b-c}{p}\sqrt{-3}$ in (2) and reducing, we have

$$\alpha^2=\frac{p}{2}(a^2-ab-ac+bc)+\frac{b^2-2bc+c^2}{p^2}\,,$$

which will not give a rational value to α unless $p=2$.

Let us assume then $\sqrt{-\beta}=\frac{b-c}{2}\sqrt{-3}$, substitute in (2) and reduce. We find $\alpha=\pm\left(a-\frac{b+c}{2}\right)$, and by substitution in (1), $\alpha=a-\frac{b+c}{2}$, and similarly for the other factors of (r). Hence

$u = \frac{1}{3}\left(a - \frac{b+c}{2} + \frac{b-c}{2}\sqrt{-3}\right)$, $\frac{1}{3}\left(b - \frac{a+c}{2} + \frac{a-c}{2}\sqrt{-3}\right)$, and $\frac{1}{3}\left(c - \frac{a+b}{2} + \frac{a-b}{2}\sqrt{-3}\right)$; similarly

$v = \frac{1}{3}\left(a - \frac{b+c}{2} - \frac{b-c}{2}\sqrt{-3}\right)$, $\frac{1}{3}\left(b - \frac{a+c}{2} - \frac{a-c}{2}\sqrt{-3}\right)$, and $\frac{1}{3}\left(c - \frac{a+b}{2} - \frac{a-b}{2}\sqrt{-3}\right)$;

$y = \frac{1}{3}(2a - b - c)$, $\frac{1}{3}(2b - a - c)$, and $\frac{1}{3}(2c - a - b)$; $x = a, b,$ and c. The three values of u are connected, as they should be, by the ratios

$$1 : \frac{1}{2}(-1 + \sqrt{-3}) : \frac{1}{2}(-1 - \sqrt{-3}).$$

If two of the roots of the equation are equal, as b and c, then $(r) = 0$ and $-\frac{1}{2}n = (a-b)^3$; and if two are imaginary, as b and c, $(= \gamma \pm \delta\sqrt{-3})$, (r) becomes rational and $-\frac{1}{2}n + (r) = (a - \gamma - 3\delta)^3$, showing why, in these two cases, Cardan's formula gives a rational result.

Numerical Equations.

1. $x^3 - 9x^2 + 14x + 24 = 0$.

Reduce the roots by 3; then $y^3 - 13y + 12 = 0$, and by Cardan's formula,

$$y = \sqrt[3]{-6 + \frac{35}{3}\sqrt{-\frac{1}{3}}} + \sqrt[3]{-6 - \frac{35}{3}\sqrt{-\frac{1}{3}}};$$

(1) $\alpha(\alpha^2 - 3\beta) = -6$, (2) $\sqrt{-\beta}\,(3\alpha^2 - \beta) = \frac{35}{3}\sqrt{-\frac{1}{3}}$;

$\sqrt{-\beta} = \sqrt{-\frac{1}{3}}$, $\frac{5}{2}\sqrt{-\frac{1}{3}}$, and $-\frac{7}{2}\sqrt{-\frac{1}{3}}$; $\beta = \frac{1}{3}$, $\frac{25}{12}$, and $\frac{49}{12}$; then by (2), $\alpha = \pm 2$, $\pm\frac{3}{2}$, and $\pm\frac{1}{2}$; and by (1), $\alpha = -2$, $\frac{3}{2}$, and $\frac{1}{2}$.

$\therefore y = -2 + \sqrt{-\frac{1}{3}} + \left(-2 - \sqrt{-\frac{1}{3}}\right) = -4$, $y = \frac{3}{2} + \frac{5}{2}\sqrt{-\frac{1}{3}} + \frac{3}{2} - \frac{5}{2}\sqrt{-\frac{1}{3}} = 3$, and $y = \frac{1}{2} - \frac{7}{2}\sqrt{-\frac{1}{3}} + \frac{1}{2} + \frac{7}{2}\sqrt{-\frac{1}{3}} = 1$; $x = -1$, 6, and 4.

2. $x^3 - 16x^2 + 73x - 90 = 0$.

Reduce the roots by $\frac{16}{3}$; then $y^3 - \frac{37}{3}y - \frac{110}{27} = 0$, and by Cardan's formula,

$$y = \frac{1}{3}\sqrt[3]{55 + 126\sqrt{-3}} + \frac{1}{3}\sqrt[3]{55 - 126\sqrt{-3}}.$$

Since $\frac{126}{27}\sqrt{-3} = \frac{3}{2}(a-b)(b-c)(a-c)\sqrt{-3}$, the factors are probably $\pm 2\sqrt{-3}$, $\pm\frac{3}{2}\sqrt{-3}$ and $\pm\frac{7}{2}\sqrt{-3}$. Hence

$$(1)\quad \alpha(\alpha^2 - 3\beta) = 55, \qquad (2)\quad \sqrt{-\beta}\,(3\alpha^2 - \beta) = 126\sqrt{-3};$$

$$\sqrt{-\beta} = 2\sqrt{-3};\quad 3\alpha^2 = 75,\quad \alpha = \pm 5;\quad \alpha = -5;$$

$$\sqrt{-\beta} = \frac{3}{2}\sqrt{-3};\quad 3\alpha^2 = \frac{363}{4},\quad \alpha = \pm\frac{11}{2};\quad \alpha = \frac{11}{2};$$

$$\sqrt{-\beta} = -\frac{7}{2}\sqrt{-3};\quad 3\alpha^2 = \frac{3}{4},\quad \alpha = \pm\frac{1}{2};\quad \alpha = -\frac{1}{2};$$

$$y = \frac{1}{3}(-5 + 2\sqrt{-3}) + \frac{1}{3}(-5 - 2\sqrt{-3}) = -\frac{10}{3},$$

$$y = \frac{1}{3}\left(\frac{11}{2} + \frac{3}{2}\sqrt{-3}\right) + \frac{1}{3}\left(\frac{11}{2} - \frac{3}{2}\sqrt{-3}\right) = \frac{11}{3},$$

$$y = \frac{1}{3}\left(-\frac{1}{2} - \frac{7}{2}\sqrt{-3}\right) + \frac{1}{3}\left(-\frac{1}{2} + \frac{7}{2}\sqrt{-3}\right) = -\frac{1}{3};$$

$$x = 2,\ 9,\ \text{and } 5.$$

V.

An Extension of Taylor's Theorem

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$$f(x+a) - f(x) = \int_0^a da \frac{d}{dx} f(x+a),\quad \left(1 - \int_0^a da \frac{d}{dx}\right) f(x+a) = f(x),$$

$$\therefore f(x+a) = \left(1 - \int_0^a da \frac{d}{dx}\right)^{-1} f(x). \qquad \text{I.}$$

$$\left(1 - \int_0^a da \frac{d}{dx}\right)^{-1} \int_0^a da\, f(x+b) = \int_0^a da \left(1 - \int_0^a da \frac{d}{dx}\right)^{-1} f(x+b)$$

(by I,)
$$= \int_0^a da \left\{1 - \int_0^{a+b} d(a+b) \frac{d}{dx}\right\}^{-1} f(x). \qquad \text{II.}$$

Expanding by I, but modifying each remainder by II before proceeding to obtain the term next following, we get

$$f(x+a+b+c+e+\&\text{c.}) = f(x+b+c+e+\&\text{c.}) + \int_0^a da\, f'(x+c+e+\&\text{c.})$$
$$+ \int_0^a da \int_0^{a+b} d(a+b)\, f''(x+e+\&\text{c.})$$
$$+ \int_0^a da \int_0^{a+b} d(a+b) \int_0^{a+b+c} d(a+b+c)\, f'''(x+\&\text{c.}) + \&\text{c.}, \qquad \text{III.}$$

which is the extension proposed.